PHYSICAL REVIEW E VOLUME 52, NUMBER 1 JULY 1995

Distribution of the first return time in fractional Brownian motion and its application to the study of on-off intermittency

Mingzhou Ding and Weiming Yang
Program in Complex Systems and Brain Sciences, Center for Complex Systems and Department of Mathematics, Florida Atlantic University, Boca Raton, Florida 33431
(Received 17 January 1995)

Herein, the term fractional Brownian motion is used to refer to a class of random walks with long-range correlated steps where the mean square displacement of the walker at large time t is proportional to t^{2H} with $0<H<1$. For ordinary Brownian motion we obtain $H=\frac{1}{2}$. Let T denote the time at which the random walker starting at the origin first returns to the origin. The purpose of this paper is to show that the probability distribution of T scales with T as $P(T)\sim T^{H-2}$. Theoretical arguments and numerical simulations are presented to support the result. Additional issues explored include modification to the power law distribution when the random walk is biased and the application of the result to the characterization of on-off intermittency, a recently proposed mechanism for bursting.

PACS number(s): 05.40.+j

I. INTRODUCTION

Consider a random walk defined as the sum [1],

$$R_n=\xi_1+\xi_2+\cdots+\xi_n \ , \tag{1}$$

where the steps ξ_i are taken from a discrete Gaussian process with $\langle \xi_i \rangle=0$ and $\langle \xi_i^2 \rangle=\sigma^2$, $\langle \ \rangle$ denoting the ensemble average. Let $C(s)=\langle \xi_i \xi_{i+s} \rangle$ be the correlation function between two steps separated by a time s. One can show [2] that if $C(s)$ decays sufficiently fast with increasing s such that $\sum_{s=0}^{\infty}C(s)$ is finite, then, for large n, the mean square displacement of R_n generally scales with n as

$$\langle R_n^2 \rangle \sim n \ . \tag{2}$$

Ordinary Brownian motion with independent steps belongs to this category. As a consequence, random walks of the above type are said to be in the Brownian domain of attraction [2].

On the other hand, if $\sum_{s=0}^{\infty}C(s)$ converges to a certain specific value and $\sum_{s=0}^{n}sC(s)$ diverges with n, or if the rate of decay of $C(s)$ is slower than that required for $\sum_{s=0}^{\infty}C(s)$ to remain finite, Eq. (2) generalizes to

$$\langle R_n^2 \rangle \sim n^{2H} \ , \tag{3}$$

for large n, where H is the Hurst exponent satisfying $0<H<1$, and the resulting random walk is called the fractional Brownian motion [3–5].

The theoretical paradigm of random walk has important applications in a variety of scientific disciplines, including physics [6], chemistry [6], and biology [7,8], as well as in engineering [9]. For example, a diffusion process can be modeled by a random walk. In particular, long-range correlated random walks with $H\neq\frac{1}{2}$ are also called anomalous diffusions [10]. Furthermore, recent studies suggest that the ideas of fractional Brownian motion provide a natural theoretical framework in which to analyze and model many phenomena arising in physics [10] and biology [8,11]. These findings are in addition to the original discovery by Hurst on long-range correlation in river flood records [12] that inspired the idea of fractional Brownian motion [3].

In the present work, we investigate the distribution [13] of the first return time T defined by the event [14]

$$R_0=0, R_1>0, R_2>0, \ldots, R_T>0 \text{ and } R_{T+1}\leq 0 \ ;$$

or, symmetrically,

$$R_0=0, R_1<0, R_2<0, \ldots, R_T<0 \text{ and } R_{T+1}\geq 0 \ .$$

Here the initial condition $R_0=0$ is added to indicate that the origin is the starting point. Our main results are as follows. First, the distribution $P(T)$ scales with T as a power law,

$$P(T)\sim T^{H-2} \ . \tag{4}$$

Second, if the random step has a finite mean $\langle \xi_i \rangle=\lambda$, i.e., the random walk is biased, the power law (4) breaks down from above at T_s where

$$T_s \sim \left[\frac{\lambda}{\sigma}\right]^{1/(H-1)} \ . \tag{5}$$

Here σ is the standard deviation of ξ_i. Third, Eq. (4) also holds for deterministic anomalous diffusions produced by chaotic processes where the steps are not Gaussian distributed. Fourth, the distribution of the first passage time appears to obey the same power law, Eq. (4), for large T. Fifth, we apply the results above to the characterization of on-off intermittency [15], a problem which originally motivated our interest in the first return time. We show that, for random modulations generated by fractional Brownian motion (fractional noise), distinct characteristics emerge.

II. THEORY AND NUMERICAL EXPERIMENTS

In this section we begin by exploiting the fractal properties of random walks to derive Eq. (4). Consider a realization of Eq. (1). Let B be a very large integer denoting its duration. If we connect the steps taken by the walker in the R versus n plane we obtain a random graph that is a fractal with a dimension of $D_1=2-H$ [4]. Imagine drawing a horizontal line at $R=0$. Due to translational symmetry in time, the length of the interval between two successive intersections between the line and the random graph approximates the first return time T. (Intuitively, the error due to the approximation should get smaller for smaller σ. Moreover, for a given σ, the relative error becomes smaller for larger T). (If we consider the continuous-time limit of the random walk, then this is no longer an approximation and it is accurate. See Sec. IV for further discussion on this point.) The intersection points again form a fractal set the dimension of which is [16]

$$D_2=D_1+1-2 \\ =1-H ,$$

where 1 in the first line is the dimension of the $R=0$ line and 2 is the dimension of the space in which the fractal is embedded. Consider the measurement of this dimension with the box-counting method [17] and let $T\gg 1$ be the box size. Then the number of boxes needed to cover the intersections is

$$N(T)\sim \frac{B}{T}-\frac{\sum_{L>T}P(L)L}{T} ,$$

where the first term is the total number of boxes needed to cover the entire interval $[0,B]$, and the second term is the total number of boxes falling in the intervals between intersections. Here for convenience we suppose that $P(L)$ is the number of intervals of length L. From the definition of the box dimension we have

$$N(T)\sim T^{-(1-H)} .$$

Combining the two equations yields

$$\frac{B}{T}-\frac{\sum_{L>T}P(L)L}{T}\sim T^{-(1-H)} ,$$

or equivalently,

$$1-\sum_{L>T}P(L)\frac{L}{B}\sim T^{H} .$$

Introducing a new variable $Y=L/B$, for large B, the sum can be replaced by an integral, giving

$$1-B\int_{T/B}P(BY)YdY\sim T^{H} ,$$

where the upper limit is some constant independent of T. Differentiating both sides with respect to T leads to

$$P(T)\sim T^{H-2} ,$$

which is Eq. (4).

We perform numerical experiments to verify Eq. (4). Random graphs, generated using the random midpoint displacement method [18], for $H=0.8$ and $H=0.3$ are displayed in Figs. 1(a) and 1(b). We fix the standard deviation of the Gaussian steps ξ_i to be $\sigma=1$ for both cases. Segments of large scale, roughly unidirectional excursions are seen in the random walk with $H=0.8$, but the random walk with $H=0.3$ appears to exhibit no such tendency. This is an important distinction that can be explained by the persistent and antipersistent trends in random walks with $H>\frac{1}{2}$ and $H<\frac{1}{2}$ [4]. By persistent we mean that a positive step tends to follow a positive step. With long-range correlation it can be further said that a generally increasing segment tends to follow a generally increasing one. For antipersistent walks the opposite is true. In Figs. 2(a) and 2(b) the distributions of T for $H=0.8$ and $H=0.3$ are plotted on a log-log scale. The straight lines, with slopes of $H-2=-1.2$ and $H-2=-1.7$, respectively, are drawn to demonstrate the good agreement with the prediction by Eq. (4).

Strictly speaking, the power law in Eq. (4) holds true for larger T only when the random walk is unbiased, i.e., $\langle \xi_i \rangle=0$. When $\langle \xi_i \rangle=\lambda\neq 0$, the distribution modifies to be

$$P(T)\sim T^{H-2}e^{-T/T_s} , \tag{6}$$

giving rise to the exponential shoulders seen in Fig. 1(b) for $\lambda=0.01$ (squares) and $\lambda=0.1$ (triangles). The power law breaks down at roughly $T=T_s$ where T_s is the time at which the systematic drift due to the bias, $T_s\lambda$, and the diffusive spread, $T_s^H\sigma$, become comparable, namely,

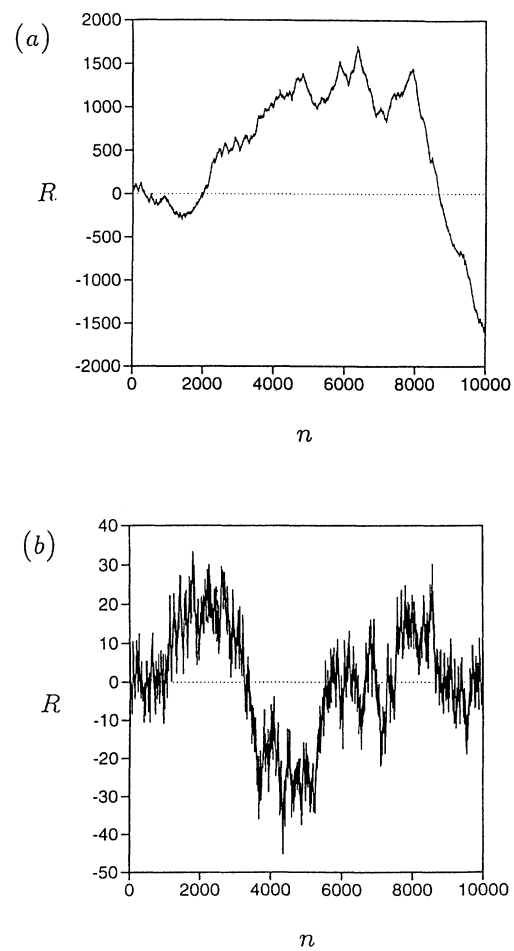

FIG. 1. Random walk realizations for (a) $H=0.8$ and (b) $H=0.3$. Here $\lambda=\langle \xi \rangle=0$ and $\sigma^2=\langle \xi^2 \rangle=1$.

$$T_s \lambda \sim T_s^H \sigma \ .$$

This can be solved to give

$$T_s \sim \left[\frac{\lambda}{\sigma} \right]^{1/(H-1)} ,$$

which is Eq. (5), where σ is the standard deviation of the random variable ξ_i. In Fig. 3 we test this relation for $H=0.3$. The straight line fit has the predicted slope of $1/(H-1)=-10/7$. Again good agreement between theory and experiment is attained. We mention that the function form in Eq. (6) is established empirically and the values of T_s used in Fig. 3 are obtained from data fitting.

Anomalous diffusions also occur in deterministic systems with chaotic dynamics and they have been used to explain the $1/f$ type of power spectra observed in Josephson junctions [19,20]. Our interest is to see whether Eq. (4) holds for this class of deterministic fractional Brownian motion. Consider one-dimensional maps of the form $R_{n+1}=g(R_n)$ where $g(R+k)=g(R)+k$ and $g(-R)=-g(R)$. For concreteness, we choose

$$g(R)=\begin{cases} R-1+a(R-m)^z \ , & m \le R \le m+\frac{1}{2} \ , \\ R+1-a(m+1-R)^z \ , & m+\frac{1}{2} \le R \le m+1 \ , \end{cases} \tag{7}$$

where m is an integer and the exponent z specifies the properties of the diffusion. In particular, the mean square displacement $\langle R_n^2 \rangle$ is expressed as [20]

$$\langle R_n^2 \rangle \sim \begin{cases} n^2 \ , & z \ge 2 \ , \\ n^{3-1/(z-1)} \ , & \frac{3}{2} < z < 2 \ , \\ n \ , & 1 < z < \frac{3}{2} \ . \end{cases}$$

In terms of the Hurst exponent, this translates to $H=1$ for $z \ge 2$, $H=\frac{3}{2}-1/[2(z-1)]$ for $\frac{3}{2}<z<2$ and $H=\frac{1}{2}$ for $1<z<\frac{3}{2}$. Note that in this type of model only persistent and ordinary diffusions $(H \ge 1/2)$ are possible. To obtain antipersistent diffusion a different class of maps exhibiting type III intermittency needs to be used [19]. Our numerical result on the distribution of the first return time is shown in Fig. 4 for $z=1.6$ and hence $H=\frac{2}{3}$. The straight line in the figure has a slope of $H-2=-4/3$. Evidently the data are in good agreement with the prediction of Eq. (4).

The purpose of this deterministic example is twofold. First, it demonstrates the validity of Eq. (4) for the distribution of the first return time in fractional Brownian motions generated by chaotic processes. Second, it generalizes the applicability of Eq. (4) to non-Gaussian processes, since unlike the calculations done earlier, the deterministic walk here has steps $\xi_i=R_i-R_{i-1}$ that are not Gaussian distributed.

Next we examine the distribution of the first passage time. By first passage time we mean the amount of time it takes for the random walker, starting at the origin, to

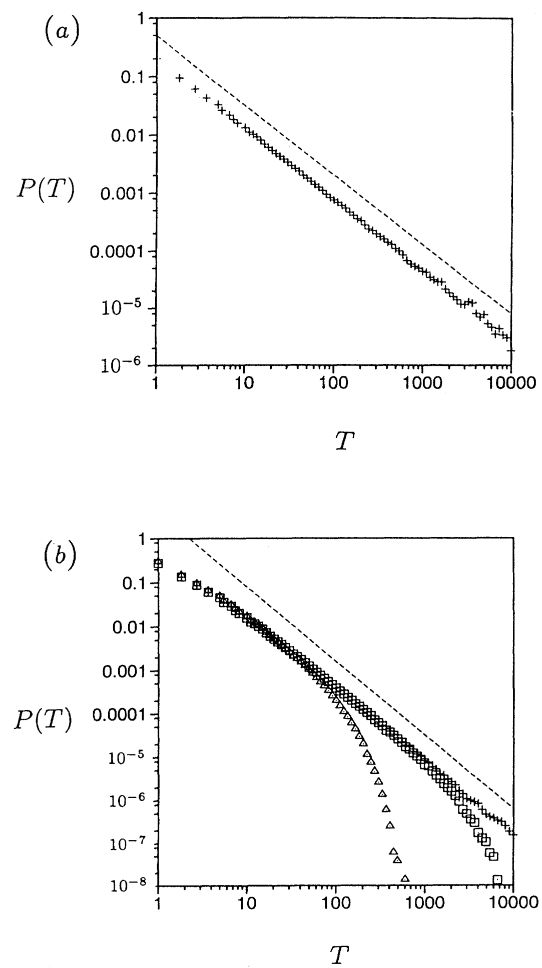

FIG. 2. Distribution $P(T)$ of the first return time T for (a) $H=0.8$ and (b) $H=0.3$. Here $\lambda=0$ in the case of (a) and $\sigma=1$ for both cases. The values of λ used in (b) are $\lambda=0.0$ (plus signs), $\lambda=0.01$ (squares) and $\lambda=0.1$ (triangles). The straight lines have slopes of (a) -1.2 and (b) -1.7, respectively.

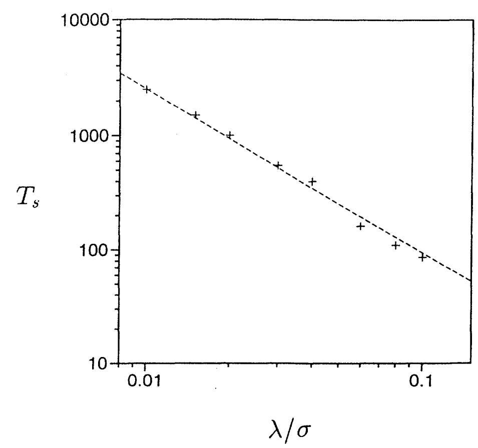

FIG. 3. The breakdown point of the power law T_s vs λ/σ computed for the type of plot in Fig. 2(b) with $H=0.3$. The slope of the straight line fit is $-\frac{10}{7}$.

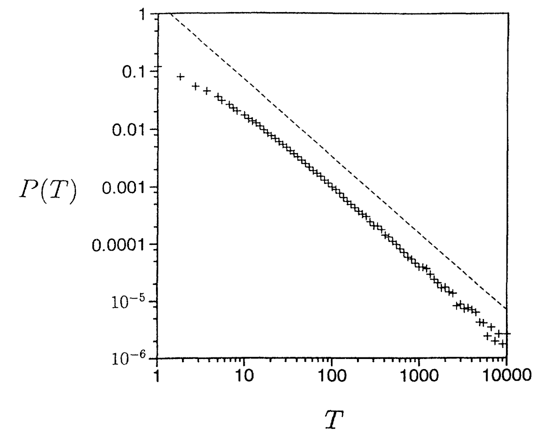

FIG. 4. Distribution $P(T)$ of the first return time T for deterministic anomalous diffusion from Eq. (7) with $z=1.6$. The slope of the straight line is $-\frac{4}{3}$.

cross a certain level for the first time. The first return time can be thought of as a special case of setting the level at zero. Based on the analysis above we conjecture that the same power law, Eq. (4), governs the distribution of the first passage time T for large T. Figure 5 shows a log-log plot of $P(T)$ versus of T for a random fractional Brownian motion with $H=0.3$ where $\lambda=0$ and $\sigma=1$. The crossing level is set at $R=1$. The slope of the straight line is $H-2=-1.7$, as would be predicted by Eq. (4), which apparently agrees with the linear trend in the data points.

III. CHARACTERIZATION OF ON-OFF INTERMITTENCY MODULATED BY FRACTIONAL NOISE

The term intermittency denotes irregular alternations in certain system variables between segments of relatively constant motion (laminar phase) and segments of erratic burst. The distinction between on-off intermittency and the three types of intermittencies of Pomeau and Manneville [21] is that during the laminar phase (*off* state), while the relevant variable in the Pomeau-Manneville case passes monotonically through a "channel," the on-off intermittent observable undergoes modulations by random or chaotic processes. The essential features of on-off intermittency are captured by the following one-dimensional map [22]:

$$x_{n+1}=z_n f(x_n) \tag{8}$$

with $f(0)=0$, $f'(0)=1$, and $z_n=a\eta_n \geq 0$ a random or a chaotic process. Given $\{\eta_n\}$, there is a critical value of a denoted a_c such that for $a<a_c$ the origin is asymptotically stable. If a is increased slightly above a_c the variable x exhibits on-off intermittency which is illustrated in Fig. 6(a) for a specific choice of $\{\eta_n\}$ and $f(x)$ (see below). Let τ be the threshold value of x above which the signal

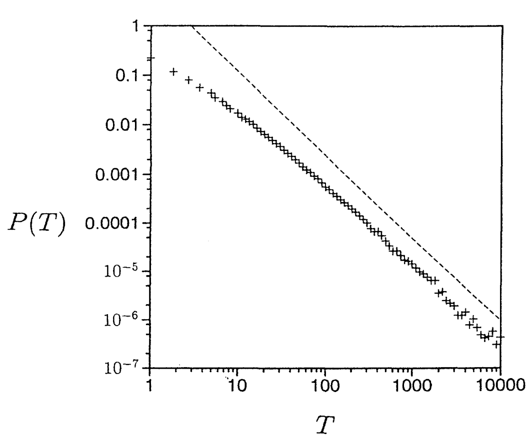

FIG. 5. Distribution $P(T)$ of the first passage time T for $H=0.3$. Here $\lambda=0$ and $\sigma=1$. The crossing level is set at $R=1.0$. The slope of the straight line is -1.7.

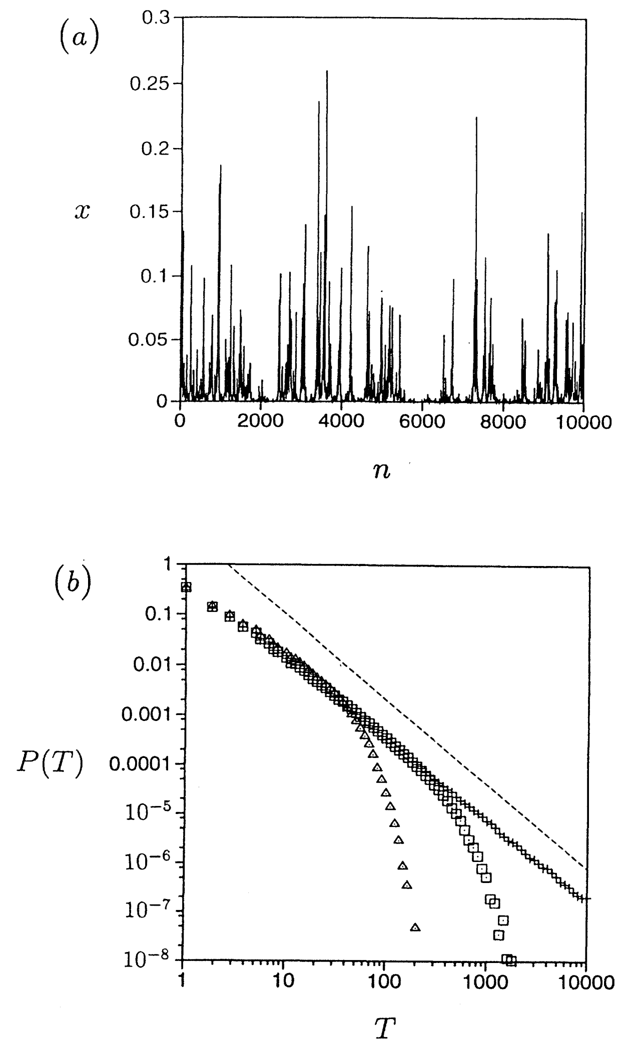

FIG. 6. (a) On-off intermittent signal x versus n from Eq. (16) with $a=1.01=a_c+0.01$. The fractional modulation noise are generated by a fractional Brownian motion with $H=0.3$. (b) Distribution $P(T)$ of the laminar phase duration T for the type of signal in (a). See text for the values of other parameters.

is considered on and below which the signal is considered off. In practice, the choice of τ should be such that for $x<\tau$ Eq. (8) can be approximated by its linearization near the origin,

$$x_{n+1}=z_n x_n \ . \tag{9}$$

The generality of Eq. (8) lies in that the linear equation (9) is a model for a diverse set of situations [23]. The nonlinear part of $f(x)$ serves to keep x bounded and reinjected back to the neighborhood of $x=0$. Its detailed form does not affect the universal features exhibited by on-off intermittent systems.

A variety of quantities have been applied to characterize signals like the one in Fig. 6(a) [15,22–24]. In this work we are mainly interested in the distribution of the laminar phase duration [22] which seems to be the most robust in an experimental situation. The problem can be analyzed as follows. Take the logarithm of both sides of Eq. (9). The result is

$$\ln x_{n+1}=\ln z_n+\ln x_n \ . \tag{10}$$

Let $s_n=\ln x_n$, $\xi_n=\ln z_n=\lambda+\ln\eta_n$, where $\lambda=\ln a$, and $R_n=\sum_{k=1}^{n}\xi_k$. Then the evolution of s can be written as

$$s_{n+1}=s_1+R_n \ , \tag{11}$$

or in terms of x,

$$x_{n+1}=e^{R_n}x_1=e^{\xi_1+\xi_2+\xi_3+\cdots+\xi_n}x_1 \tag{12}$$

which is described by the random walk (or chaotic walk if the underlying modulation is chaotic [22])

$$R_n=\xi_1+\xi_2+\xi_3+\cdots+\xi_n \ , \tag{13}$$

with the step size ξ_n and $R_0=0$. On-off intermittency occurs if $\langle\xi_n\rangle$ is greater than zero. It will not occur if $\langle\xi_n\rangle$ is less than zero. When $\langle\xi_n\rangle=0$, meaning that the random walk is unbiased, we have the critical case. The value of a_c is found to be

$$\lambda_c=\ln a_c=-\langle\ln\eta\rangle \ . \tag{14}$$

We henceforth take $\langle\ln\eta\rangle=0$ and thus $a_c=1$.

Assume, without loss of generality, that at the beginning of each laminar episode, $x_1=\tau$. The occurrence of a laminar phase of length T is then determined by

$$R_0=0,\ R_1<0,\ R_2<0,\ R_3<0,\ldots,R_{T-1}<0\ , \quad \text{and } R_T\geq 0 \ . \tag{15}$$

Clearly, this quantity T is the same as the first return time defined in Sec. I. Note that in the present context the return of interest occurs when the walker passes the zero axis from below. The distribution of the corresponding T, however, is the same as that of the first return time involving returns from both directions. We thus use the same variable T to denote both the first return time and the laminar phase duration.

Past work on on-off intermittency has only considered noise in the Brownian domain of attraction. In particular, regarding the distribution $P(T)$ of the laminar phase duration T, it is shown in [22] that

$$P(T)\sim T^{-3/2} \ ,$$

corresponding to letting $H=\frac{1}{2}$ in Eq. (4). In what follows we study the effect of fractional noise [3,25] modulation (see next section for further discussions of this aspect). For numerical work we choose Eq. (8) to be

$$x_{n+1}=z_n x_n e^{-bx_n} \ , \tag{16}$$

where $0\leq x<\infty$ and $b>0$. When $z_n=a>1$ ($\eta_n\equiv 1$) is a constant, the fixed point near the origin is found to be $x^*=\ln a/b$. The value of x^* can be used to give a rough estimate of the threshold τ below which we have the laminar phase and a linear approximation for Eq. (16) is valid. By adjusting b we can vary the relative size of the linear region. This feature gives the map more flexibility than, say, the logistic map. The fractional noise ξ_i is generated as follows. Create the path of a random walk R_n with a given H using the algorithm of random midpoint displacement utilizing steps from a Gaussian distribution with zero mean and a variance of σ^2. From R_n compute the realization sequence $\ln\eta_i$ according to $\ln\eta_i=R_i-R_{i-1}$. Then, by definition, $\xi_i=\ln a+\ln\eta_i$, and the modulation z_i is obtained as $z_i=e^{\xi_i}$. The critical value for a is $a_c=1$. By choosing $b=1.0$, $\sigma=0.4$, and $H=0.3$ the system responses for $a=a_c+0.01$ are shown in Fig. 6(a). If $a<a_c$ the values of x rapidly decay to near zero and stay there for as long as the experiment is operating. Using $\tau=0.1$ as the threshold value we compute the distribution of laminar phase durations for the type of signal in Fig. 6(a) and the result is shown in Fig. 6(b). The parameter values used are $b=0.01$, $\sigma=0.05$, and $H=0.3$. The plus signs are for $a=a_c+0.00005$, the squares for $a=a_c+0.005$ and the triangles for $a=a_c+0.05$. The straight line in the figure has a slope of $H-2=-1.7$ and it agrees well with the data points. This result is to be expected from the discussion preceding and about Eq. (15) that the distribution $P(T)$ of the laminar phase duration T be the same as that of the first return time, namely, Eq. (4). In the same figure we also observe the breakdown of the power law from above when $a>a_c=1$. The correction term in this case is again exponential as in Eq. (6). The breakdown point T_s has the scaling relation,

$$T_s\sim\left[\frac{\lambda}{\sigma}\right]^{1/(H-1)} \ ,$$

where $\lambda=\ln a$.

IV. SUMMARY AND DISCUSSION

The central result of this paper is Eq. (4), which states that the distribution $P(T)$ of the first return time T in fractional Brownian motion scales with T as a power law, where the exponent is a function of the Hurst exponent H. A theoretical argument based on the fractal nature of random walks is employed to derive the relation. Further support from numerical experiments is presented. When the random walk is biased, the power law breaks down from above, and an exponential correction term needs to be introduced. Additionally, we demonstrate that the same power law form holds for the distribution of the first passage time, and moreover, it holds for the distribution of the first return time in anomalous diffusions generated by deterministic chaotic processes, where the steps are not Gaussian distributed. This latter point leads us to conjecture that for a given random walk, the asymptotic form of the first return time distribution depends only on the mean square displacement at large time, Eq. (3), not on the distribution of the random walk steps. This seems understandable since Eq. (3) specifies the self-affine properties of the underlying fractal which, in turn, provide the basis for the argument used in Sec. II to derive Eq. (4). Finally, we find that the characters of on-off intermittent signals with fractional noise modulation are indexed by the Hurst exponent, which include the case with ordinary Brownian noise as a special case.

To conclude, we make two remarks, one on the relevance of this work to the case of continuous-time fractional Brownian motion, and the other on the motivation, from the viewpoint of a stochastic differential equation, behind the form of fractional noise perturbation, e^{ξ_i}, used in the equations of Sec. III.

Strictly speaking, for a continuous-time fractional Brownian motion, the distribution of the first return time is not well defined in the sense that a finite time interval may contain infinite many crossing points with the zero axis [26]. In particular, from Eq. (4) we see that the limit $T \to 0$ is singular and the distribution is not normalizable. On the other hand, if we introduce a lower cutoff value for the interval between successive crossings and take into account only such interval whose length is longer than the cutoff value, it would still be possible to define the time of first return and consider its distribution. This stipulation is reasonable from an experimental point of view since any measurement apparatus will contain a natural cutoff due to finite precision. We thus believe that the results in this paper are also applicable to the continuous-time case provided the above restriction is properly implemented.

Next, let us consider the Langevin equation [27],

$$\frac{dx_t}{dt} = (\lambda - \beta x_t^2 + \xi_t) x_t \ , \tag{17}$$

where $x_t \geq 0$, $\beta > 0$, and ξ_t is a multiplicative Gaussian process of zero mean $\langle \xi_i \rangle = 0$. Clearly, $x_t \equiv 0$ is a solution of Eq. (17), and its stability is determined by the linearization about the origin

$$\frac{dx_t}{dt} = (\lambda + \xi_t) x_t \ , \tag{18}$$

where $|x_t| \ll 1$. One can formally solve Eq. (18) and obtain

$$x_t = e^{R_t} x_0 = e^{\int_0^t (\lambda + \xi_s) ds} x_0 \ . \tag{19}$$

If $\lambda < 0$ then the origin is stable with probability one. The critical case occurs at $\lambda = 0$. When $0 < \lambda \ll 1$ it is shown in [27] that, for Gaussian white noise ξ_t, Eq. (17) exhibits on-off intermittency.

The same stability analysis and conclusion apply when ξ_i is a fractional noise process with long-range correlation. In this case the characteristics of the on-off intermittent signal will be a function of the Hurst exponent H defined by the mean square displacement relation

$$\langle (R_t - \lambda t)^2 \rangle \sim t^{2H} \ . \tag{20}$$

Let τ be a small number so that $x_t < \tau$ is in the off state. The evolution of such x_t is approximately governed by Eq. (18). The distribution of the laminar phase duration depends on the behavior of the continuous-time fractional Brownian motion $R_t = \int_0^t (\lambda + \xi_s) ds$ in Eq. (19). It is the analogy between Eq. (12) and Eq. (19) that constitutes the basis, from a differential equation point of view, for considering fractional noise in the equations in Sec. III in the form of e^{ξ_i}. Further analogies can be made between (8) and (17), and between (9) and (18). These analogies lead us to expect that many other universal properties of on-off intermittency be the same for both discrete and continuous systems. This, together with the relative easiness of handling discrete maps, is seen as further motivation for the study carried out in Sec. III. We will report our result on the study of Eq. (17) with fractional noise in a future publication [28].

ACKNOWLEDGMENTS

We would like to thank H. Hastings, J. Heagy, J.-P. Kahane, B. Mandelbrot, C.-K. Peng, M. Shlesinger, and R. Voss for communications and discussions. This work was supported by a grant from the Office of Naval Research.

[1] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1968 and 1971), Vols. I and II.

[2] B. B. Mandelbrot and J. R. Wallis, Water Res. Res. **4**, 909 (1968).

[3] B. B. Mandelbrot and J. W. Van Ness, SIAM Rev. **10**, 422 (1968).

[4] B. B. Mandelbrot, *Fractal Geometry of Nature* (Freeman, New York, 1983).

[5] Conventionally, Brownian motion, ordinary [1] or fractional [3], is a term used to denote continuous-time stochastic processes. A random walk can be regarded its discrete analogue. Since both types of processes share many similarities we use the same designation to avoid the introduction of additional terminology.

[6] H. Haken, *Synergetics* (Springer-Verlag, New York, 1983); G. H. Weiss and R. J. Rubin, Adv. Chem. Phys. **52**, 363 (1983); C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, New York, 1990).

[7] H. C. Berg, *Random Walks in Biology* (Princeton University, Princeton, 1983).

[8] J. B. Bassingthwaighte, L. S. Liebovitch, and B. J. West, *Fractal Physiology* (Oxford University Press, Oxford, 1994).

[9] Y. K. Lin, *Probabilistic Theory of Structural Dynamics* (Krieger, Malabar, FL, 1976).

[10] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990). For a discussion on anomalous diffusions associated with the dynamics of Lévy flights see M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature **363**, 31 (1993).
[11] For a sample, see C.-K. Peng *et al.*, Nature **356**, 168 (1992); Phys. Rev. Lett. **70**, 1343 (1993); R. F. Voss, *ibid.* **68**, 3805 (1992); A. Schenkel, J. Zhang, and Y. C. Zhang, Fractals **1**, 47 (1993); J. J. Collins, Phys. Rev. Lett. **73**, 764 (1994); K. A. Selz *et al.*, Fractals (to be published).
[12] H. E. Hurst, Trans. Am. Soc. Civil Eng. **116**, 770 (1951).
[13] In this paper the term distribution means probability density function.
[14] In a more strict sense, this event should be called first zero crossing, rather than first return, given the discrete-time and continuous-space nature of the problem. But, since the term zero crossing does not take the initial state into consideration, we will still refer to T as the first return time and ignore the fine distinction.
[15] A. S. Pikovsky, Z. Phys. B **55**, 149 (1984); H. Fujsaka and T. Yamada, Prog. Theor. Phys. **74**, 918 (1985); L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990); Physica D **53**, 102 (1991); N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993).
[16] K. J. Falconer, *The Geometry of Fractal Sets* (Cambridge University Press, Cambridge, 1986).
[17] See, for example, E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
[18] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Chaos and Fractals* (Springer-Verlag, New York, 1992). The original random midpoint displacement method for ordinary Brownian motion was introduced by A. Fournier, D. Fussell, and L. Carpenter, in Comm. ACM **25**, 371 (1982).
[19] T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).
[20] T. Geisel, J. Nierwetberg, and A. Zacherl, Phys. Rev. Lett. **54**, 616 (1985); M. F. Shlesinger and J. Klafter, *ibid.* **54**, 2551 (1985).
[21] Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).
[22] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994); N. Platt, S. M. Hammel, and J. F. Heagy, Phys. Rev. Lett. **72**, 3498 (1994); P. W. Hammer *et al.*, Phys. Rev. Lett. **73**, 1095 (1994).
[23] W. Yang and M. Ding (unpublished).
[24] A. S. Pikovsky, Phys. Lett. A **165**, 33 (1992); T. Yamada and H. Fujisaka, Prog. Theor. Phys. **76**, 582 (1986); H. Fujisaka *et al.*, *ibid.* **76**, 1198 (1986).
[25] J. Feder, *Fractals* (Plenum, New York, 1988).
[26] We thank Steve Lowen for discussions on this point.
[27] T. Yamada and H. Fujisaka, Prog. Theor. Phys. **84**, 824 (1990); H. Fujisaka and T. Yamada, *ibid.* **90**, 529 (1993).
[28] M. Ding and W. Yang (unpublished).